Resonance between Noise and Delay

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We propose here a stochastic binary element whose transition rate depends on its state at a fixed interval in the past. With this delayed stochastic transition this is one of the simplest dynamical models under the influence of "noise" and "delay". We demonstrate numerically and analytically that we can observe resonant phenomena between the oscillatory behavior due to noise and that due to delay.

Resonant behavior is one of the most studied and utilized fundamental physical phenomena. As well as being of interest in a variety of fields of physics, ranging from elementary particle experiments, such as resonance analyzed with the Breit–Wigner formula [1] to resonant electrical circuits [2], it has recently been actively investigated in the context of biological information processing as an application of stochastic resonance (see, e.g., [3–6]). In these studies, noise, which is normally considered an obstacle to information processing, is treated as an enhancer of information processing through its resonance with external signals. It is known that delay, which is another element present and studied in biological phenomena and information processing (see, e.g., [7,8]), can cause a complex oscillatory behavior in an otherwise simple, stable dynamical system. Analogous resonant phenomena with respect to delay have also been noted and investigated. In [9], for example, the effect of delay in a lateral inhibition in a neural network is investigated both experimentally and theoretically. It is found that delayed lateral inhibition can cause amplification of neural responses to sinusoidal stimuli in spite of the fact that, without delay, these inhibition generally attenuates such responses.

The main theme of this paper is presentation of a simple model which shows that delay and noise can have a resonance between themselves without an external periodic driving force. Our model is a two state system whose dynamics is governed by combinations of its state at some fixed interval in the past and noise with a certain width. In the probability space, this model can be described as a stochastic binary element whose transition probability depends on its state at some fixed interval in the past. Thus, in this description the model can be considered as a previously studied "delayed random walk" [10] except that it can only take two states. With a fixed delay, we show analytically and numerically that such a model has a residence time histogram which shows a peak of maximum height with appropriately chosen transition probability, i.e., noise and delay "tuned" together exhibit a resonance. From the point of view of stochastic resonance, this is a new type and one of the simplest models which is analytically tractable. We conclude the report with a discussion of possible utilization of such a resonant behavior with noise and delay.

Before presenting our model, we briefly discuss a numerical study on a simple one dimensional system to illustrate the physics motivating proposal of the model. Let us consider a simple one dimensional map dynamics with noise ξ and delay τ, which is formally given by

$$z(t+1) = \tanh[\beta(z(t-\tau) - \theta)] + \xi_L \tag{1}$$

β and θ are parameters and ξ_L has the following probability distribution.

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$$P(\xi = u) = \frac{1}{2L} \quad (-L \leq u \leq L)$$
$$= 0 \quad (u < -L, u > L) \tag{2}$$

In words, ξ_L is a time uncorrelated uniformly distributed noise taking the range $(-L, L)$. This map can be considered as a discrete time correspondence of the following differential equation model.

$$\frac{dz}{dt} = -\frac{\partial}{\partial z}V(z) + \xi_L$$
$$V(z) = \frac{1}{2}z^2(t) - \frac{1}{\beta}\log[\cosh(\beta(z(t-\tau) - \theta))] \tag{3}$$

The shape of the asymmetric potential $V(z)$ with no delay is shown in Fig 1(A). We have numerically simulated the map (1) with various noise width and delay and found that we can have a regular spiking behavior as shown in Fig 1(C) for tuned noise width and delay. Also, we have observed that the signal to noise ratio of the corresponding peak in the Fourier spectrum goes through a maximum with varying noise width as is generally found in a system showing a stochastic resonance [11]. We can qualitatively argue that the delay alters the effective potential into an oscillatory one just like that due to an external oscillating force leading to a stochastic resonance with tuned noise width. The analysis of the dynamics given by (1) or (3), however, is a non–trivial task. Our model is an approximate abstraction of this dynamics retaining asymmetric stochastic transition and delay in order to gain insight into the resonant behavior between noise and delay.
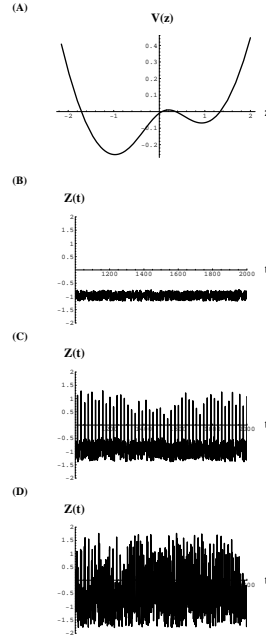

FIG. 1. (A) The shape of the asymmetric potential for $\beta = 2.0$ and $\theta = 0.1$. Also the typical dynamics of $Z(t)$ from the map model as we change noise width L. The values of L are (B) $L = 0.2$, (C) $L = 0.4$, (D) $L = 0.8$. The data is taken with $\tau = 20$, $\beta = 2.0$, $\theta = 0.1$ and the initial condition $Z(t) = 0.0$ for $t \in [-\tau, 0]$. The plots are shown between $t = 1000$ to 2000.

Let us now describe our model in detail. The state of the system $X(t)$ at time step t can take either -1 or 1. With the same noise ξ_L, we can define our model formally.

$$X(t+1) = \theta[f(X(t-\tau)) + \xi_L],$$
$$f(n) = \frac{1}{2}((a+b) + n(a-b)),$$
$$\theta[n] = 1 \quad (0 \leq n), \qquad -1 \quad (0 > n), \tag{4}$$

where a and b are parameters such that $|a| \leq L$ and $|b| \leq L$, and τ is the delay. In relation to the map (1), this model is an approximate discretization of space into two states with a and b controlling the bias of transition (reflecting the two different barrier heights from the two stationary points of the potential), depending on the state of X at τ steps before.

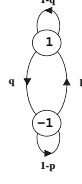This model can be described in the probability space more concisely as shown in Figure 2.

FIG. 2.   Schematic view of the stochastic model dynamics.

The formal definition is given as follows:

$$\begin{aligned}
P(1, t+1) &= p, & X(t-\tau) = -1,\\
&= 1-q, & X(t-\tau) = 1,\\
P(-1, t+1) &= q, & X(t-\tau) = 1,\\
&= 1-p, & X(t-\tau) = -1,\\
p &= \frac{1}{2}(1 + \frac{b}{L}),\\
q &= \frac{1}{2}(1 - \frac{a}{L}), & & (5)
\end{aligned}$$

where $P(s,t)$ is a probability that $X(t) = s$. Hence, the transition probability of the model depends on its state at τ steps past and is a special case of delayed random walks [10].

We first investigate the model numerically and observe that a qualitatively similar feature to those shown in Figure 1 appears. We randomly generate $X(t)$ for the interval $t = (-\tau, 0)$. Simulations are performed in which parameters are varied and $X(t)$ is recorded up to 10^6 steps. From the trajectory $X(t)$, we construct a residence time histogram $h(u)$ for the system to be in the state -1 for u consecutive steps. Some examples of histograms and corresponding $X(t)$ are shown in Figure 3 ($q = 1 - q = 0.5$, $\tau = 10$). We note that with $p << 0.5$, as in Figure 3(A), the model has a tendency to switch or spike to $X = 1$ state after the time step interval of τ. But the spike trains do not last long and result in a small peak in the histogram. For the case of Figure 3(C) where p is closer to 0.5, we observe less regular transitions and the peak height is again small. With appropriate p as in Figure 3(B), spikes tend to appear at interval τ more frequently, resulting in higher peaks in the histogram.
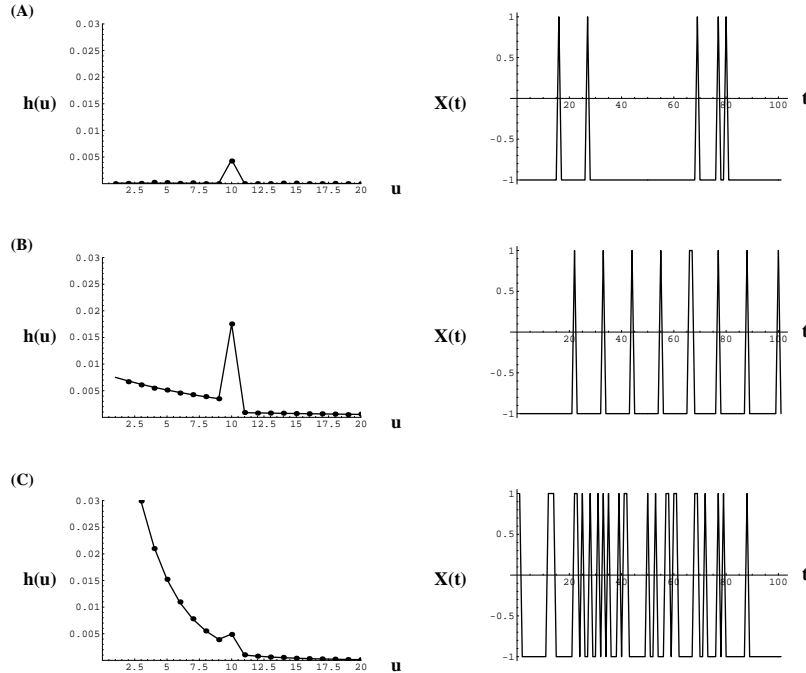

FIG. 3.   Residence time histogram and dynamics of $X(t)$ as we change p. The values of p are (A) $p = 0.005$, (B) $p = 0.05$, (C) $p = 0.2$. The solid line in the histogram is from the analytical expression given in equations (8,9,10)

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This change of peak height in histograms which reaches maximum at an appropriate noise level is one way to characterize stochastic resonance. Choosing an appropriate p is equivalent to "tuning" noise width L with other parameters appropriately fixed. In this sense, our model shares a feature of the stochastic resonance. It can be classified among models of stochastic resonance without an external signal [12]. The difference and the new point is the use of delay as a source of its oscillatory dynamics. With this characteristic, it could be termed as stochastic resonance with delayed dynamics or, equivalently, a resonance between noise and delay.

In order to make this point clearer, let us treat the model analytically. The first observation to make with the model is that given τ, it consists of statistically independent $\tau + 1$ Markov chains. Each Markov chain has its state appearing at every $\tau + 1$ interval. With this property of the model, we label time step t by the two integers s and k as follows

$$t = s(\tau + 1) + k, \quad (0 \le s, 0 \le k \le \tau) \tag{6}$$

Let $P_\pm(t) \equiv P_\pm(s, k)$ be the probability for the state to be in the ± 1 state at time t or (s, k). Then, it can be derived that

$$P_+(s, k) = \alpha(1 - \gamma^s) + \gamma^s P_+(s = 0, k),$$
$$P_-(s, k) = \beta(1 - \gamma^s) + \gamma^s P_+(s = 0, k),$$
$$\alpha = \frac{p}{p + q},$$
$$\beta = \frac{q}{p + q},$$
$$\gamma = 1 - (p + q). \tag{7}$$

In the steady state, we have $P_+(s \to \infty, k) \equiv P_+ = \alpha$ and $P_-(s \to \infty, k) \equiv P_- = \beta$. The steady state residence time histogram can be obtained by computing the following quantity, $h(u) \equiv P(+; -, u; +)$, which is the probability that the system takes consecutive -1 state u times between two $+1$ states. With the definition of the model and the property of statistical independence between Markov chains in the sequence, the following expression can be derived:

$$P(+; -, u; +) = P_+(P_-)^u P_+ = (\beta)^u (\alpha)^2 \quad (1 \le u < \tau) \tag{8}$$
$$= P_+(P_-)^\tau (1 - q) = (\beta)^\tau (\alpha)(1 - q) \quad (u = \tau) \tag{9}$$
$$= P_+(P_-)^\tau (q)(1 - p)^{u-\tau}(p) = (\beta)^u (p)^2 \quad (u > \tau) \tag{10}$$

With appropriate normalization, this expression can reflect the shape of the histogram obtained by numerical simulations as shown in Figure 3. Also, by differentiating equation (9) with respect to p, we can derive the resonant condition for the peak to reach maximum height as

$$q = p\tau \tag{11}$$

or, equivalently,

$$L - a = (L + b)\tau. \tag{12}$$

Figure 3 shows how the maximum height changes with p and τ. We see that peak maximum is reached by choosing parameters according to equation (11). Also, by changing q, we can control the width and height of graphs in Figure 4. An example is shown in Figure 5. We note that this analysis for the histogram is exact in the stationary limit, which is another feature of the model.
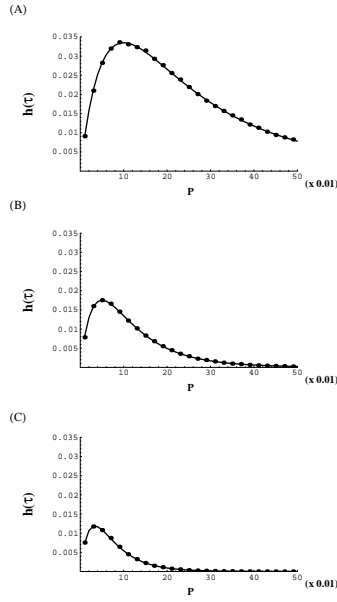
FIG. 4.   A plot of peak height by varying p The solid line is from equation (9).
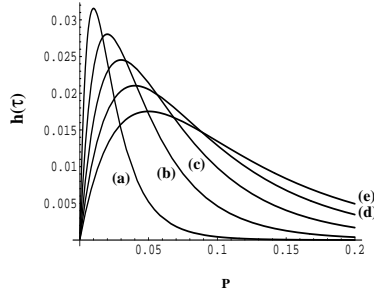

FIG. 5.   Resonance curves with various q from equation (9). The parameters are $\tau = 10$, and $q =$ (a)0.1, (b)0.2, (c)0.3, (d)0.4, (e)0.5.

We have proposed here a very simple model which nonetheless illustrates resonance behavior between noise and delay both numerically and analytically. By relating a and b to the membrane threshold, the model could be used as a very simplified and abstracted model of spiking neuron with delayed self–feedback [13], and could be developed for a model of pulse coupled neurons with delay [14]. Also, with the analytically tractable resonant characteristics of the model described in this paper, we could possibly seek an application of this delayed stochastic binary element for information processing. For example, one may be able to code temporal information stochastically, or an encryption scheme might be developed by extending this model [15]. Exploration of such applications as well as extension of the model into many body systems are the focus of our current research.

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